APPENDIX A.

THE DE RHAM-WITT COMPLEX AND ADDITIVE CHOW
GROUPS OVER A FIELD: THE CHARACTERISTIC 2 CASE

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The main theorems of [Rü07b] (see also [Rü07a]) were stated only for fields of characteristic $\neq 2$. This originates in the use of [HM04, Thm 4.2.8] which was only for odd primes. In this appendix it is explained that thanks to [Cos08] the results of [Rü07b] extend directly to the characteristic 2 case. I thank Amalendu Krishna and Jinhyun Park for the opportunity to detail this extension here.

A.1. Let $(\mathbb{W}_S^{\infty})_S$ denote the big de Rham Witt complex from [Hes15], where $A$ is a ring and $S$ is running through all truncation sets. It comes with the maps $d$, $F_n$, $V_n$, restriction, and multiplication. For $m \geq 1$ we set $\mathbb{W}_m^{\infty} \Omega^* := \mathbb{W}_{\{1,\ldots,m\}}^{\infty} \Omega^*$. Note that in general $d \circ d$ is not zero in $\mathbb{W}_S^{\infty} \Omega^*$. Denote by $\mathbb{W}_S^{\infty} / \mathbb{Z}$ the Witt complex from [Hes15, Rem 4.8] (with $k = \mathbb{W}(\mathbb{Z})$); it is the initial object in the category of Witt complexes with $\mathbb{W}(\mathbb{Z})$-linear differential. Note that $\mathbb{W}_S^{\infty} / \mathbb{Z}$ is always a dga, in particular we have $d \circ d = 0$. Furthermore, if $A$ contains a field, then $\mathbb{W}_S^{\infty} / \mathbb{Z} = \mathbb{W}_S^{\infty}$ see [Hes15, Rem 4.2, c)]; if $A$ is an $\mathbb{F}_p$-algebra, then the $p$-typical de Rham-Witt complex is the one from Bloch-Deligne-Illusie. Also note in case $A$ is an $\mathbb{F}_p$-algebra or a $\mathbb{Q}$-algebra, then we have a decomposition

\begin{equation}
\mathbb{W}_S^{\infty} \Omega^* = \prod_{(j,p) = 1} \mathbb{W}_{P \cap S/j}^{\infty} \Omega^*,
\end{equation}

as in [Rü07b, Thm 1.11]. (Indeed, in this case the construction of the $V$-complex in [Rü07b, Prop 1.2] goes through and the same proof as in [Rü07b, Thm 1.11] shows that it is the initial object in the category of Witt complexes as in [Hes15] and that it decomposes as in (A.1.1).)

**Theorem A.2** ([Rü07b, Thm 3.20] for char($k$) $\neq 2$). Let $k$ be a field. Then there is an isomorphism

$$
\mathbb{W}_m^{\infty} \Omega^* \cong \text{CH}^{n-1}(\mathbb{A}_k^n \setminus \{0\}, n), \quad m \geq 1,
$$

where the right hand side is the additive Chow groups of Bloch-Esnault. Furthermore, via this isomorphism, the maps $d$, $F_n$, $V_n$, restriction and multiplication on the de Rham-Witt side correspond to $D$, $F_n$, $V_n$, restriction, and $*$, on the Chow side, as defined in [Rü07b, Def-Prop 3.9].

Thanks to [Cos08], which was not at disposal when [Rü07b] was written, the proof of *loc. cit.* goes through, also for $p = 2$. We will explain this in more detail in the following.

A.3. We fix a prime $p$. Let $A$ be a $\mathbb{Z}_{(p)}$-algebra and denote by $A[x]$ the polynomial ring in the variable $x$. Then the group $W_n^{\infty} \Omega^*_A[x]/\mathbb{Z}$ (resp. $W_n^{\infty} \Omega^*_A[x,1/x]/\mathbb{Z}$) is freely

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generated by elements of the following type:
\[ a[x]^j, \quad a \in W_n \Omega^q_{A/Z}, \ j \geq 0 \text{ (resp. } j \in \mathbb{Z}), \]
\[ b[x]^j d[x], \quad b \in W_n \Omega^{q-1}_{A/Z}, \ j \geq 1 \text{ (resp. } j \in \mathbb{Z}), \]
\[ V^s (a[x]^j), \quad a \in W_n - \Omega^q_{A/Z}, \ s \in \{1, \ldots, n - 1\}, \ j \geq 1 \text{ with } (j, p) = 1 \]
\[ (\text{resp. } j \in \mathbb{Z} \setminus p\mathbb{Z}), \]
\[ dV^s (b[x]), \quad b \in W_n - \Omega^{q-1}_{A/Z}, \ s \in \{1, \ldots, n - 1\}, \ j \geq 1 \text{ with } (j, p) = 1 \]
\[ (\text{resp. } j \in \mathbb{Z} \setminus p\mathbb{Z}). \]

For \( A[x] \) and \( p \) odd this is [HM04, Thm 4.2.8], for \( p = 2 \) this follows from [Cos08, Thm 4.3] (one has to observe that the functor \( \Lambda \) constructed in this references sends a \( \mathbb{W}(\mathbb{Z}) \)-linear p-typical Witt complexes over \( A \) to a \( \mathbb{W}(\mathbb{Z}) \)-linear p-typical Witt complex over \( A[x] \) and that it preserves surjections); the result for \( A[x, 1/x] \) is deduced from this as in [Rüll07b, Thm 2.1], where the reference to [Rüll07b, Prop 1.18] should be replaced by [Hes15, Thm C].

**Theorem A.4** ([Rüll07b, Thm 2.6] for \( \text{char}(k) \neq 2 \)). Let \( L/k \) be a finite field extension. Then there exists a trace map
\[ \text{Tr}_{L/k} : \mathbb{W}_S \Omega_L^* \to \mathbb{W}_S \Omega_k^*, \]
which satisfies the properties (i) - (v) from [Rüll07b, Thm 2.6], furthermore [Rüll07b, Prop 2.7] holds.

**Proof.** The proof is the same as in loc. cit., once we made the following remarks: [Rüll07b, Lem 1.20] holds for any \( F_p \)-algebra \( A \) with the same proof; [Rüll07b, Lem 2.3] also holds for \( p = 2 \), this follows from [Ill79, I, Prop 3.2, 3.4] and a limit argument: [Rüll07b, Prop 2.4] holds with the same proof also for \( p = 2 \) once the reference to [Rüll07b, Thm 2.1] is replaced by A.3 above. \( \square \)

**A.5.** Let \( p \) be a prime and \( A \) a \( \mathbb{Z}_p \)-algebra. For a finite truncation set \( S \) we define
\[ \text{Fil}_{S,j} := \ker (\mathbb{W}_S \Omega^*_A / (t)/\mathbb{Z} \to \mathbb{W}_S \Omega^*_A / (t)/(v)/\mathbb{Z}), \ j \geq 1, \]
and
\[ \mathbb{W}_S \Omega^*_A / (t)/\mathbb{Z} = \lim_{\rightarrow j} \mathbb{W}_S \Omega^*_A / (t)/\mathbb{Z} / \text{Fil}_{S,j}. \]

Then any element in \( \mathbb{W}_S \Omega^*_A / (t)/\mathbb{Z} \) can be uniquely written as in [Rüll07b, (2.9.1)] and we can define
\[ \text{Res}^q_{t,n} : W_n \Omega^q_A / (t)/\mathbb{Z} \to W_n \Omega^{q-1}_A /\mathbb{Z} \]
as in [Rüll07b, (2.9.2)]. (Using A.3, the proof is similar as in [Rüll07b, Lem 2.9].)

We define \( \text{Res}^q_{t,n} \) as the composition
\[ W_n \Omega^q_A / (t)/\mathbb{Z} \xrightarrow{\text{can}} W_n \Omega^q_A / (t)/\mathbb{Z} \xrightarrow{\text{Res}^q_{t,n}} W_n \Omega^{q-1}_A /\mathbb{Z}, \]
and if \( A \) contains a field and \( S \) is a finite truncation set, then we define
\[ \text{Res}^q_{t,S} : \mathbb{W}_S \Omega^q_A / (t) \to \mathbb{W}_S \Omega^{q-1}_A \]
as in [Rüll07b, Def 2.11], using that in this case we have \( \mathbb{W}_S \Omega^*_A = \mathbb{W}_S \Omega^*_A /\mathbb{Z} \) and that the decomposition (A.1.1) also extends to \( \mathbb{W}_S \Omega^* \).
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For any \( \mathbb{Z}(p) \)-algebra \( A \) the map \( \text{Res}^q_{t,n} \) satisfies the properties (i) - (viii) of [Rül07b, Prop 2.12] and [Rül07b, Lem 2.14] holds; if \( A \) contains a field the same holds for \( \text{Res}^q_{t,S} \), \( S \) any finite truncation set. (The case \( \text{Res}^q_{t,n} \) is proven as in loc. cit., the case \( \text{Res}^q_{t,S} \) follows from this. Note however, that even in the case where \( A \) contains a field, the proof of property (iv) and of Lemma 2.14 uses the reduction to a torsion free \( \mathbb{Z}(p) \)-algebra. Since for \( p = 2 \) the absolute de Rham-Witt complex is not a dga, we prefer to work with the \( \mathbb{W}(\mathbb{Z}) \)-linear complex.)

Remark A.6. For an \( \mathbb{F}_p \)-algebra \( A \), the residue \( \text{Res}_t : W_n^q\Omega^q_{A(t)} \to W_n^q\Omega^q_{A}^{p-1} \) was also constructed in [Kat80, §2, Prop 3] using algebraic K-theory and Bloch’s approach to the de Rham-Witt complex.

Theorem A.7 ([Rül07a, Thm 2] for \( \text{char}(k) \neq 2 \)). Let \( C \) be a connected regular projective curve over a field \( k \) with function field \( K = k(C) \). Let \( S \) be a finite truncation set. Then

\[
\sum_{P \in C} \text{Res}_P(\omega) = 0, 
\text{for all } \omega \in W_S^q\Omega^q_K, \quad q \geq 1,
\]

where the sum is over all closed points in \( C \) and \( \text{Res}_P : W_S^q\Omega^q_K \to W_S^q\Omega^q_k^{p-1} \) is defined as in [Rül07a, Def-Prop 1] (using \( \text{Res}_{t,S} \) from A.5 and \( \text{Tr} \) from A.4.)

Proof. The same proof of [Rül07a, Thm 2], [Rül07b, Thm 2.19] goes through once we made the following remarks: the proof of the well-definedness of \( \text{Res}_P \) is the same as in [Rül07b, Def-Prop 2.15] since [Rül07b, Lem 1.16] holds in general; [Rül07b, Prop 2.18] holds with the trace from Theorem A.4; at the end of the proof of [Rül07b, Thm 2.19] (on page 140/141) an element is lifted to \( W_S^1\Omega^1_A((t))/\mathbb{Z} \) with \( A = \mathbb{Z}(p)[a, b, c] \), replace this by the following argument (at least if \( p = 2 \)): first observe that the looked for vanishing can be reduced to the \( p \)-typical case by the definition of \( \text{Res}_P \); then lift the element \( \omega_{2,P} \) to the \( W(\mathbb{Z}) \)-linear complex \( W_n^q\Omega^q_{A((t))}/\mathbb{Z} \) and proceed as in the proof using the \( \text{Res}_t \) from A.5. \( \square \)

Proof of Theorem A.2. The proof of [Rül07b, Thm 3.20] (see also [Rül07a]) goes through once we made the following remarks: in [Rül07b, Lem 3.5] replace \( W_m^r\Omega^r_A \) by \( W_m^r\Omega^r_{A/Z} \) (at least if \( p = 2 \)); at the end of the proof of [Rül07b, Thm 3.16] (on page 148) refer to Theorem A.7 instead of [Rül07b, Thm 2.19]; in [Rül07b, Lem 3.15] observe that if \( \text{char}(k) = 2 \), then \( \mathcal{D}D(\alpha) = 0 \), since in \( K^M_2(k) \) we have \( \{a, a\} = \{-1, a\} = 0 \), and similar also \( \mathcal{F}_r, \mathcal{D}_r = \mathcal{D} \); this implies that [Rül07b, Prop 3.17] also holds if \( \text{char}(k) = 2 \); for the rest of the proof of [Rül07b, Thm 3.20] use A.4 for properties of the trace and A.3 instead of [Rül07b, Thm 2.1]. \( \square \)

References


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